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ABSTRACT

In the technique developed by K. G. Joreskog to solve the problem for oblique rotation to a specified simple structure, the basic concept is that the simple structure solution itself is determined only by the zero coefficients of the reference-structure matrix and not by the coefficients of non-zero magnitude. Following this, prior information about the desired simple structure is taken into account to impose zeros on some of the factor loadings. This way an n x r target matrix H is built with the zero elements specified and the others unspecified. This specific Procrustean rotation involves r eigenproblems. When prior information for the desired solution is not available, additional efforts are required to construct a target. A new strategy is proposed for this purpose, applying the technique of vector majorization. The constructed target matrix H has the same form as in Joreskog's work, but with all elements specified (both zeros and non-zeros), which transforms Joreskog's specific Procrustean rotation into a normal Procrustean rotation that enables the application of any well-known procedure for Procrustean rotation, and thus, avoiding the eigenproblems. A slightly different problem can also be solved when only the number of zeros in the target is known. All computational examples are based on 24 psychological tests of Holzinger and Harman. There are seven tables of example data. (SLD)

Vector Majorization Technique for Rotation to a Specified Simple Structure[1]

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About twenty years ago Jöreskog developed original technique to solve the problem for oblique rotation to a specified simple structure. The basic concept is that the simple structure solution itself is determined only by the zero coefficients of the reference-structure matrix and not by the coefficients of nonzero magnitude. Following this, he take into account the prior information about the desired simple structure to impose zeros on some of the factor loadings. This way an $n \times r$ target matrix H is build with the zero elements specified and the others unspecified. This specific Procrustean rotation involvs r eigenproblems.

When prior information for the desired solution is not available, additional efforts are required to construct a target. In the present work new strategy is proposed for this purpose, applying the technique of vector majorization. The constructed target matrix H has the same form as in Jöreskog's work, but with *all elements specified* (both zeros and nonzeros). That simply means we transform Jöreskog's specific Procrustean problem into normal Procrustean one which enable us to apply any of well known procedures for Procrustean rotation, avoiding the eigenproblems. Moreover slightly different problem is solved, when the *number* of zeros in the target are known only. All computational examples are based on Holzinger & Harman 24 psychological tests.

Key words: Target matrix construction, vector majorization, Procrustean rotation.

The purpose of the work is to demonstrate the application of the vector majorization technique for the simple structure solution study.

We shall start with a brief introduction in vector majorization theory. Vector majorization is defined in a number of different ways in Marshall & Olkin (1979), but the most intuitive one is the following: The vector X is majorized from the vector Y when the components of Y are more distinguishable, more non - uniform than the components of X, in which case we write $X \prec Y$. For example:

$$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \prec (\frac{1}{2}, \frac{1}{2}, 0) \prec (1, 0, 0).$$

For clarity we introduce a strong mathematical:

Definition 1. (Marshall & Olkin, 1979) Let X , Y be n - dimentional vectors. The vector Y is said to majorize another vector X and it is denoted by $X \prec Y$ if the inequalities

$$\sum_{i=1}^{k} X_{[i]} \leq \sum_{i=1}^{k} Y_{[i]}$$

hold for each $k = 1, 2, \ldots, n-1$, where $X_{[1]} \geq X_{[2]} \geq, \ldots, \geq X_{[n]}$ denote the components of X in decreasing order and the equality

$$\sum_{i=1}^{n} X_{[i]} = \sum_{i=1}^{n} Y_{[i]}$$

holds.

Let us consider the problem for rotation to already specified simple structure. Let A represent the initial $n \times r$ orthogonal factor matrix. Let H denote the hypothetical $n \times r$ pattern matrix of some target simple sructure of the form:

$$H = \begin{bmatrix} x & 0 & 0 \\ x & 0 & 0 \\ 0 & x & 0 \\ 0 & x & x \\ x & x & 0 \\ 0 & 0 & x \\ 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \\ x & 0 & 0 \end{bmatrix},$$

with the zero elements specified and the others (represented by "x") unspecified. Jöreskog (1965), see also Mulaik (1972), developed an original technique, involving r eigenproblems, to obtain an $r \times r$ transformation matrix T with $\mathrm{diag}(T^T T) - I_r = 0_r$, such that AT fits the zero elements of H in the least squares sense.

It is easy to consruct a target H when a prior information about the desired factor solution is available. But often we do not have such information. Usually to obtain the target we do, first, PROMAX transformation of the initial orthogonal factor matrix A in order to localize the zeros in the simple solution. Denoting PROMAX target by $A' = \{a'_{ij}\}$, we define the target $H = \{h_{ij}\}$ as follows:

$$h_{ij} = \begin{cases} 0 & , a'_{ij} \in (\alpha_j, \beta_j) \\ "x" & , otherwise \end{cases} ,$$

where the intervals $(\alpha_j, \beta_j), j = 1, 2, \ldots, r$ are determined more or less in some subjective, intuitive manner. After that the application of Jöreskog's method is straightforward. Quite often, in practice, the researchers put unities instead of "x" in the target H. So they do not make use Jöreskog's zero - fitting method, but simply the standard Procrustean rotations.

An application of the above consideration is given in Mulaik (1972) , where is obtained factor solution for Holzinger & Harman 24 psychological tests using Jöreskog's zero - fitting method. As a starting point is taken VARIMAX rotated solution (Mulaik, 1972, p. 264) . Because the desired simple structure is unknown, in Mulaik (1972) is proposed the following procedure, in order to obtain target matrix for Jöreskog's zero - fitting method. First PROMAX transform of the initial VARIMAX matrix is performed. For convenience, both of them are displied in Table 1.

Insert Table 1 about here.

As a second step all elements in already obtained PROMAX target which are less than .060 are replaced by .000 . The elements which are equal or greater than .060 remain undetermined and are denote by "x" in so called Jöreskog's type target. This target, the obtaned oblique solution and the correlations among corresponding primary factors are taken from Mulaik (1972, p. 318) and they are given for convenience in Table 2 and Table 3 respectively.

Insert Table 2 and Table 3 about here.

Let us now return to vector majorization. Let X and $C = (c, c, \ldots, c)$ be n - dimensional vectors and define the vector:

$$(X - C)^0 = (max(X_1 - c, 0), max(X_2 - c, 0), \ldots, max(X_n - c, 0)).$$

Hereafter we shall use for any vector X the notation

$$\rho(X) = \left(\frac{X_1}{\sum_{i=1}^n X_i}, \frac{X_2}{\sum_{i=1}^n X_i}, \ldots, \frac{X_n}{\sum_{i=1}^n X_{ij}} \right)^T .$$

Proposition 1. (Marshall & Olkin, 1979) If $X = (X_1, X_2, \ldots, X_n)$ and $X_i > 0$, $i = 1, 2, \ldots, n$, the relation

$$\rho(X) \prec \rho((X - C)^0)$$

holds for any $c < max(X_1, X_2, \ldots, X_n)$.

Now we already have a theoretical basis for constructing the target matrix. Let $\vec{a_j}$ be j-th column vector of some $n \times r$ matrix A. In order to majorized it, making its components more " non - uniform ", let us form:

$$(a_{1j}^2, a_{2j}^2, \ldots, a_{nj}^2)^T$$

and apply **Proposition 1** to it. Then we have

$$\left(\frac{a_{1j}^2}{\sum_{i=1}^n a_{ij}^2}, \frac{a_{2j}^2}{\sum_{i=1}^n a_{ij}^2}, \ldots, \frac{a_{nj}^2}{\sum_{i=1}^n a_{ij}^2} \right)^T \prec$$

$$\left(\frac{max(a_{1j}^2 - c_j^2, 0)}{\sum_{i=1}^n max(a_{ij}^2 - c_j^2, 0)}, \ldots, \frac{max(a_{nj}^2 - c_j^2, 0)}{\sum_{i=1}^n max(a_{ij}^2 - c_j^2, 0)} \right)^T$$

for any $c_j^2 < max(a_{1j}^2, a_{2j}^2, \ldots, a_{nj}^2)$.

When the matrix A is the initial $n \times r$ orthogonal factor matrix, the components of the former vector are known as relative contribution of the j-th factor. This terminology will be kept further for any A . If we compose the sequence of absolute values of the components of $\vec{a_j}$ in increasing order $|a_{[1]j}| \leq |a_{[2]j}| \leq \ldots \leq |a_{[n]j}|$ and let us choose the number of the components to retain nonzero, say k_j. Then for any $c_j \in [|a_{[n-k_j]j}|, |a_{[n-k_j+1]j}|)$ the latter vector majorizes the former of its relative contributions in " $\prec$ " sense and

has exactly k_j nonzero components, called *new* relative contributions. The corresponding vector, containing the new contributions of the j-th factor has the form:

$$(max(a_{1j}^2 - c_j^2, 0), \ldots, max(a_{nj}^2 - c_j^2, 0))^T ,$$

with also just k_j nonzero elements. Then we form a new target matrix with the following elements:

$$\mathrm{sign}(a_{ij})\sqrt{\max(a_{ij}^2 - c_j^2, 0)} .$$

It is possible to prove that the best choice for c_j from the vector majorization point cf view is $|a_{[n-k_j]j}|$.

Performing this procedure to the PROMAX target from Table 1 with $k_1 = 8$, $k_2 = 6$, $k_3 = 11$ and $k_4 = 9$ new target will be obtained. It is given in Table 4 and has zeros in the same places as in so called Jöreskog's type target, but with all elements specified. Then we are able to apply any of well known procedures for Procrustean rotation (both orthogonal and oblique). The corresponding oblique solution is in the same Table. In Table 5 the correlation among primary factors are given.

Insert Table 4 and Table 5 about here.

These two solutions are quite close. This conclusion follows from the corresponding correlations among primary factors also. The obtained new factors are a bit more oblique that Mulaik's ones. The difference between the solutions in least - square sence is .06204.

More intriguing is to perform previous procedure to VARIMAX solution from Table 1 with the same k_1, k_2, k_3 and k_4 . Actually this problem is different from Mulaik's one, because the zero positions are not fixed a priori. So they are not suposed to be the same, but as we shall see the result is quite similar to Mulaik's one, being more orthogonal. The least - square difference between these two solutions is .06356. The corresponding new target, oblique solution and correlations among factors are given in Table 6 and Table 7 respectively.

Insert Table 6 and Table 7 about here.

It has been demonstrated a new point of view for studying the simple structure concept. It was considered already solved numerical example and

it was shown that new solution is adequate and close to the original one. Although Jöreskog's method gives an optimal solution in the least - square sense, the presented approach has own merits making possible to search for both oblique and orthogonal solution and being simpler computationally.

References

Jöreskog, K. G. (1965) On rotation to a specified simple structure. *Res. Bul.* 65 - 13. Princeton, N. J.: Educ. Test. Serv.

Marshall, A. W., and Olkin, I. (1979) *Inequalities: Theory of Majorization and Its Applications.* Academic Press.

Mulaik, S. A. (1972) *The Foundations of Factor Analysis.* New York: McGraw - Hill .

Table 1.

	VARIMAX solution				PROMAX target			
#	I	II	III	IV	I	II	III	IV
1	.140	.190	.670	.170	.001	.005	.708	.003
2	.100	.070	.430	.100	.002	.000	.778	.002
3	.150	.020	.540	.080	.005	.000	.826	.000
4	.200	.090	.540	.070	.013	.000	.716	.000
5	.750	.210	.220	.130	.701	.004	.005	.001
6	.750	.100	.230	.210	.706	.000	.006	.004
7	.820	.160	.210	.080	.807	.001	.003	.000
8	.540	.260	.380	.120	.317	.017	.078	.001
9	.800	.010	.220	.250	.726	.000	.004	.007
10	.150	.700	-.060	.240	.001	.730	.000	.010
11	.170	.600	.080	.360	.003	.470	.000	.061
12	.020	.690	.230	.110	.000	.773	.009	.000
13	.180	.590	.410	.060	.003	.397	.098	.000
14	.220	.160	.040	.500	.022	.006	.000	.589
15	.120	.070	.140	.500	.002	.000	.005	.749
16	.080	.100	.410	.430	.000	.001	.072	.250
17	.140	.180	.060	.640	.002	.005	.000	.775
18	.000	.260	.320	.540	.000	.021	.049	.399
19	.130	.150	.240	.390	.005	.008	.053	.373
20	.350	.110	.470	.250	.086	.001	.279	.022
21	.150	.380	.420	.260	.003	.124	.184	.027
22	.360	.040	.410	.360	.091	.000	.154	.091
23	.350	.210	.570	.220	.051	.007	.362	.008
24	.340	.440	.220	.340	.060	.167	.010	.060

Table 2.

	Jöreskog - type target				Mulaik's solution			
#	I	II	III	IV	I	II	III	IV
1	.000	.000	x	.000	-.077	.060	.734	.018
2	.000	.000	x	.000	-.026	-.022	.474	.013
3	.000	.000	x	.000	.017	-.099	.606	-.029
4	.000	.000	x	.000	.069	-.018	.591	-.073
5	x	.000	.000	.000	.805	.107	.021	-.093
6	x	.000	.000	.000	.804	-.045	.019	.041
7	x	.000	.000	.000	.913	.056	.007	-.157
8	x	.000	x	.000	.496	.164	.276	-.103
9	x	.000	.000	.000	.875	-.166	-.012	.109
10	.000	x	.000	.000	.041	.765	-.227	.090
11	.000	x	.000	x	.018	.590	-.086	.246
12	.000	x	.000	.000	-.171	.753	.204	-.099
13	.000	x	x	.000	.003	.602	.405	-.202
14	.000	.000	.000	x	.133	.033	-.159	.551
15	.000	.000	.000	x	-.004	-.080	-.002	.581
16	.000	.000	x	x	-.118	-.066	.361	.442
17	.000	.000	.000	x	-.003	.026	-.153	.741
18	.000	.000	.000	x	-.240	.118	.230	.573
19	.000	.000	.000	x	-.009	.026	.144	.395
20	x	.000	x	.000	.234	-.044	.412	.135
21	.000	x	x	.000	-.044	.302	.383	.123
22	x	.000	x	x	.252	-.146	.309	.307
23	.000	.000	x	.000	.197	.067	.538	.050
24	x	x	.000	x	.219	.367	.058	.264

Table 3.

	Correlations among primary factors			
#	I	II	III	IV
I	1.000	.429	.563	.535
II	.429	1.000	.425	.554
III	.563	.425	1.000	.548
IV	.535	.554	.548	1.000

Table 4.

	Target				Solution			
#	I	II	III	IV	I	II	III	IV
1	.000	.000	.706	.000	-.090	.042	.766	-.014
2	.000	.000	.776	.000	-.037	-.036	.502	-.008
3	.000	.000	.824	.000	.002	-.122	.651	-.056
4	.000	.000	.714	.000	.058	-.038	.626	-.098
5	.699	.000	.000	.000	.811	.103	-.001	-.077
6	.704	.000	.000	.000	.804	-.055	.010	.059
7	.805	.000	.000	.000	.917	.047	-.009	-.140
8	.313	.000	.057	.000	.497	.157	.266	-.104
9	.724	.000	.000	.000	.872	-.180	-.012	.131
10	.000	.730	.000	.000	.068	.806	-.335	.107
11	.000	.470	.000	.055	.037	.622	-.171	.258
12	.000	.773	.000	.000	-.153	.780	.131	-.109
13	.000	.396	.082	.000	.013	.614	.362	-.220
14	.000	.000	.000	.588	.135	.045	-.194	.570
15	.000	.000	.000	.749	-.008	-.077	-.014	.590
16	.000	.000	.049	.249	-.129	-.073	.373	.431
17	.000	.000	.000	.775	-.002	.040	-.192	.761
18	.000	.000	.000	.398	-.244	.126	.210	.568
19	.000	.000	.000	.372	-.013	.028	.134	.395
20	.069	.000	.274	.000	.225	-.059	.430	.125
21	.000	.122	.176	.000	-.043	.306	.364	.110
22	.075	.000	.145	.087	.241	-.161	.327	.304
23	.000	.000	.358	.000	.189	.052	.553	.032
24	.032	.166	.000	.054	.229	.381	.006	.212

Table 5.

	Correlations among primary factors			
#	I	II	III	IV
I	1.000	.459	.596	.538
II	.452	1.000	.538	.578
III	.596	.538	1.000	.610
IV	.538	.578	.610	1.000

Table 6.

#	Target I	Target II	Target III	Target IV	Solution I	Solution II	Solution III	Solution IV
1	.000	.000	.589	.000	-.055	.066	.714	.034
2	.000	.000	.287	.000	-.015	-.014	.462	.017
3	.000	.000	.435	.000	.028	-.086	.590	-.028
4	.000	.000	.435	.000	.080	-.011	.576	-.060
5	.669	.000	.000	.000	.776	.105	.052	-.054
6	.669	.000	.000	.000	.771	-.034	.056	.058
7	.746	.000	.000	.000	.877	.055	.040	-.118
8	.420	.000	.205	.000	.486	.159	.287	-.063
9	.724	.000	.000	.000	.835	-.145	.032	.114
10	.000	.650	.000	.000	.05[illegible]	.721	-.211	.161
11	.000	.541	.000	.249	.029	.564	-.067	.294
12	.000	.639	.000	.000	-.143	.707	.190	-.021
13	.000	.530	.256	.000	.026	.565	.388	-.126
14	.000	.000	.000	.427	.125	.050	-.120	.532
15	.000	.000	.000	.427	-.005	-.055	.028	.548
16	.000	.000	.256	.342	-.106	-.043	.370	.419
17	.000	.000	.000	.585	-.006	.048	-.110	.710
18	.000	.000	.000	.473	-.222	.132	.244	.557
19	.000	.000	.000	.291	-.005	.040	.161	.383
20	.083	.000	.344	.000	.233	-.029	.419	.139
21	.000	.277	.272	.000	-.027	.293	.380	.153
22	.118	.000	.256	.249	.245	-.121	.328	.292
23	.083	.000	.472	.000	.203	.073	.537	.069
24	.000	.355	.000	.219	.220	.355	.079	.240

Table 7.

#	Correlations among primary factors I	II	III	IV
I	1.000	.351	.498	.480
II	.351	1.000	.361	.431
III	.498	.361	1.000	.482
IV	.480	.431	.482	1.000